SYDNEY GRAMMAR SCHOOL

2013 Trial Examination

FORM VI
MATHEMATICS EXTENSION 1

Monday 12th August 2013

General Instructions
- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks
- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist
- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 113 boys

Examiner
BR/DNW

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The coefficient of x^2 in the expansion of $(2 + x)^6$ is: **1**

(A) 15

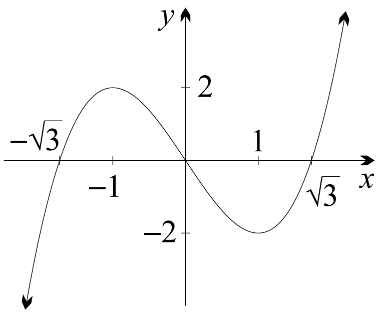
(B) 60

(C) 160

(D) 240

QUESTION TWO

1



The graph above shows $y = g(x)$ where $g(x) = x^3 - 3x$.
The inverse will be a function if the domain is restricted to:

(A) $x > 0$

(B) $-\sqrt{3} < x < \sqrt{3}$

(C) $x < -1$ or $x > 1$

(D) $x < -\sqrt{3}$ or $x > \sqrt{3}$

QUESTION THREE

Let $f(x) = x\sqrt{1 - x^2}$. The value of $\displaystyle\int_{-1}^{1} f(x)\,dx$ is: **1**

(A) 0 because $f(x)$ is odd

(B) 0 because $f(x)$ is even

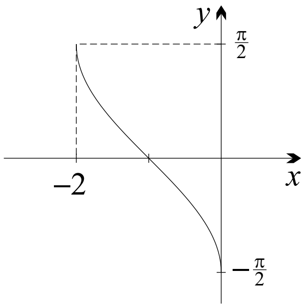
(C) $2\displaystyle\int_{0}^{1} f(x)\,dx$ because $f(x)$ is odd

(D) $2\displaystyle\int_{0}^{1} f(x)\,dx$ because $f(x)$ is even

Exam continues next page ...

QUESTION FOUR

1



The diagram above shows the graph of:

(A) $y = \sin^{-1}(x+1)$

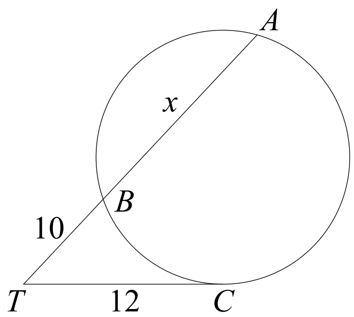
(B) $y = \sin^{-1}(x-1)$

(C) $y = \cos^{-1}(x+1) - \frac{\pi}{2}$

(D) $y = \cos^{-1}(x-1) - \frac{\pi}{2}$

QUESTION FIVE

1



In the diagram above, the tangent at C meets the secant AB at T. Given that $AB = x$, $BT = 10$ and $CT = 12$, the value of x is:

(A) 2

(B) $4\frac{2}{5}$

(C) 8

(D) $14\frac{2}{5}$

QUESTION SIX

The solution of $\dfrac{x^2-1}{x} > 0$ is: **1**

(A) $x < -1$ or $x > 1$

(B) $-1 < x < 1$ and $x \neq 0$

(C) $-1 < x < 0$ or $x > 1$

(D) $x < -1$ or $0 < x < 1$

Exam continues overleaf ...

QUESTION SEVEN

A particle exhibits simple harmonic motion according to the equation $v^2 = (x-1)(5-x)$. **1**
The amplitude is:

(A) 1

(B) 2

(C) 3

(D) 4

QUESTION EIGHT

In a kitchen, a leg of ham is removed from a fridge and its temperature H degrees Celsius is monitored. It is found that after t hours the temperature is given by $H = 16(1 - \frac{3}{4}e^{-4t})$. **1**
The temperature of the kitchen is:

(A) 3°C

(B) 4°C

(C) 10°C

(D) 16°C

QUESTION NINE

It is known that $(x+2)$ is a factor of the polynomial $P(x)$ and that **1**

$$P(x) = (x^2 + x + 1) \times Q(x) + (2x + 3)$$

for some polynomial $Q(x)$. From this information alone, it can be deduced that:

(A) $Q(-2) = -\frac{1}{3}$

(B) $Q(-2) = \frac{1}{3}$

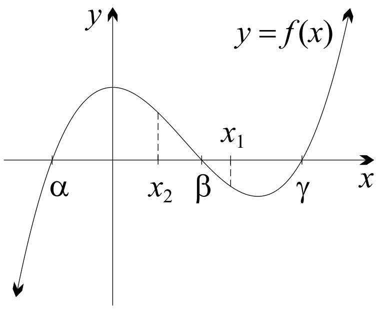
(C) $Q(2) = -1$

(D) $Q(2) = 1$

Exam continues next page ...

QUESTION TEN

1



A student is using the method of halving the interval to find the root $x = \alpha$ for the function sketched above, beginning with the approximations x_1 and x_2. Which of the following statements is correct?

(A) The method fails because $f'(0) = 0$.

(B) The method fails because $x_1 > \alpha$ and $x_2 > \alpha$.

(C) The method succeeds because $f(x_1)$ and $f(x_2)$ have opposite sign.

(D) The method succeeds because $x_1, x_2, \ldots, x_n$ is a decreasing sequence.

—— End of Section I ——

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(b) Differentiate $x \sec x$ with respect to x. **2**

(c) The line through $A(1,5)$ and $B(-1,-2)$ is divided externally in the ratio $3:1$ by the point P. Find the x-coordinate of P. **2**

(d) Find the equation of the tangent to $y = \dfrac{1}{x+2}$ at the point $P\left(2, \frac{1}{4}\right)$. **3**

Give your answer in general form.

(e) Use the substitution $u = x + 1$ to evaluate $\displaystyle\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$. **3**

(f) Find and simplify the term independent of x in the expansion of $\left(\dfrac{3x^2}{2} - \dfrac{1}{3x}\right)^9$. **3**

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $f(x) = \cos^{-1}(x) + \cos^{-1}(-x)$.

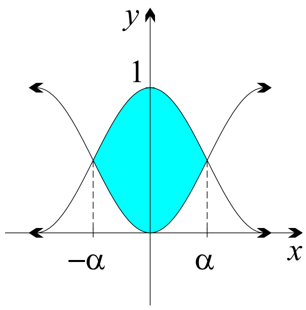
(i) Show that $f(x)$ is constant by finding $f'(x)$. **1**

(ii) Find the value of the constant. **1**

(b) The solution of $e^{-x} = x$ is approximately $x_0 = 0{\cdot}5$. Use one application of Newton's method to find another approximation to this solution. Give your answer correct to three decimal places. **3**

Exam continues next page ...

(c)

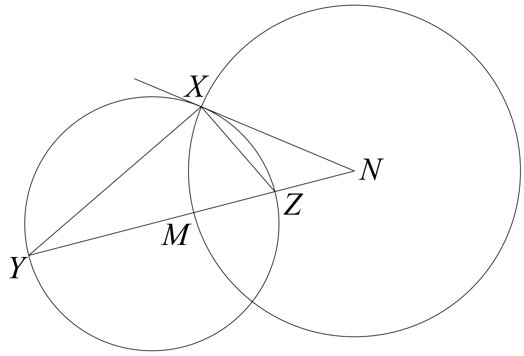


The graph above shows the curves $y = \cos^2 x$ and $y = \sin^2 x$.

(i) Find the value of α. **1**

(ii) Find the area of the shaded region in the graph. **2**

(d) **3**



The diagram above shows a tangent to a circle at X and a secant YZ in the same circle. The tangent and secant intersect at N. The circle with radius NX intersects YZ at M.

Copy the diagram into your examination booklet.

Prove that MX bisects $\angle YXZ$. Begin by letting $\angle ZXN = \alpha$ and $\angle ZXM = \beta$.

(e) The city of Mongerville has a large population M. A rumour begins and t hours later the number of people who have heard the rumour is P. It is found that the rate at which the rumour spreads is proportional to the number of people who have not heard the rumour. Thus

$$\frac{dP}{dt} = k(M - P)\,.$$

(i) Show that $P = M + Be^{-kt}$ is a solution of this equation. **1**

(ii) Assuming that no-one has heard the rumour initially, what is the value of B? **1**

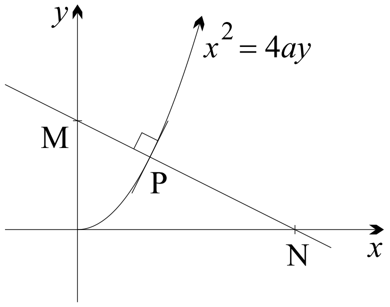
(iii) After 5 hours, half the population of Mongerville has heard the rumour. What is the value of k? **1**

(iv) How long does it take for 95% of the population to have heard the rumour? Give your answer correct to the nearest minute. **1**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The polynomial equation $4x^3 - 12x^2 + 5x + 6 = 0$ has roots α, β and γ. It is known that one of the roots is the sum of the other two. Using this fact and the results for the sum of the roots and the product of the roots, find α, β and γ. **3**

(b)



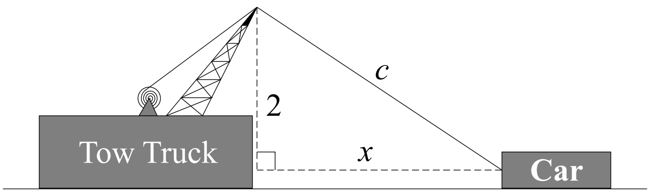
A normal is drawn to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$, where $p > 0$. This normal intersects the x-axis at N and the y-axis at M as shown in the diagram above.

(i) Find the equation of the normal PN. **2**

(ii) It is known that the point P divides NM in the ratio $3 : 2$. **2**

Find the value of the parameter p.

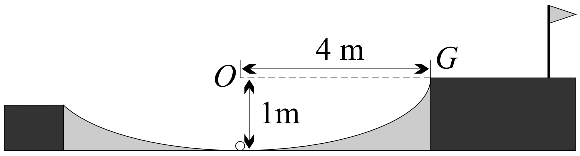
(c) **3**



A car is attached to the winch on a tow truck by a steel cable. The top of the cable is 2 metres above the point of attachment. At time t seconds the length of the cable is c metres and the winch is reducing this at the rate of $\dfrac{dc}{dt} = -0{\cdot}1\,\text{m/s}$. The horizontal distance between the car and the tow truck is x metres. The situation is shown in the schematic diagram above.

Find the rate at which the car is moving towards the tow truck when the length of the cable is $5{\cdot}2\,\text{m}$. Give your answer correct to the nearest cm/s.

(d)



The diagram above shows a golf ball in the middle of a 1 m deep bunker and 4 m from the edge of the green at G. The ball is hit with an initial speed of 12 m/s at an angle of elevation α. Putting the origin at O, the horizontal and vertical equations of motion are:

$$x = 12t\cos\alpha \qquad \text{and} \qquad y = -5t^2 + 12t\sin\alpha - 1\,.$$

(You are NOT required to prove these equations.)

(i) Find the maximum height that the ball reaches above G when $\alpha = 30^\circ$. **2**

(ii) Find the range of values that α may take so that the ball lands on the green at or beyond G. Give your answer correct to the nearest 5°. **3**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) A particle is moving in simple harmonic motion along the x-axis, with centre the origin and amplitude 5 cm. Initially the particle is at $x = -3$ cm and moving with a positive velocity $v = 12$ cm/s.

(i) Use the given information and the equation $v^2 = n^2(a^2 - x^2)$ to show that the period of the motion is $\frac{2\pi}{3}$ seconds. **1**

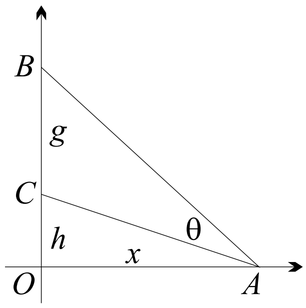
(ii) Draw a sketch of the displacement-time graph for $0 \le t \le \frac{2\pi}{3}$. **1**

(iii) The displacement-time graph has equation $x = a\sin(nt - \theta)$. **2**

Find the value of θ in radians correct to two decimal places.

(b) Given $y = \tan^{-1}\left(\dfrac{a}{x}\right)$, show that $\dfrac{dy}{dx} = \dfrac{-a}{x^2 + a^2}$. **1**

(c)



A billboard of height g metres is mounted on a wall at BC which is h metres above level ground at O. The billboard subtends an angle θ when observed at ground level at A, x metres from O. The situation is shown in the diagram above.

(i) Briefly explain why **1**

$$\theta(x) = \tan^{-1}\left(\frac{g+h}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right).$$

(ii) This function $\theta(x)$ has one stationary point where it reaches its maximum value. Show that at this point $x = \sqrt{h(g+h)}$. **2**

(iii) It is known that the maximum value is $\theta = \frac{\pi}{4}$. Show that $g = 2h(1 + \sqrt{2})$. **2**

(d) For all integers r and n, where $1 \le r \le n$, let

$$^{n}\mathcal{L}_r = {}^{n}\mathrm{C}_r \times {}^{n-1}\mathrm{C}_{r-1} \times (n-r)!$$

(i) Find and simplify expressions for $^{n}\mathcal{L}_1$ and $^{n}\mathcal{L}_n$. **2**

(ii) Show that for $2 \le r \le n$, **3**

$$^{n+1}\mathcal{L}_r = {}^{n}\mathcal{L}_r \times (n+r) + {}^{n}\mathcal{L}_{r-1}.$$

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\text{NOTE: } \ln x = \log_e x, \; x > 0$$

Multiple Choice (with possible source of errors)

Q 1 (D) (A) ${}^6\mathrm{C}_2$, (B) $2^2 \times {}^6\mathrm{C}_2$, (C) $2^3 \times {}^6\mathrm{C}_3$

Q 2 (D) (A) fails $y = -1$ (B) fails $y = 1$ (c) fails $y = 1$

Q 3 (A) (B) f is not even, (C) only if f is even, (D) f is not even

Q 4 (C) (A) reflected in x-axis, (B) reflected in y-axis, (D) shifted right

Q 5 (B) (A) $x + 10 = 12$, (C) $x(x + 10) = 12^2$, (D) $10x = 12^2$

Q 6 (C) (A) $x^2 - 1 > 0$ (B) $x^2 - 1 < 0$ (D) $\frac{x^2-1}{x} < 0$

Q 7 (B) (A) x_{min} (C) centre of motion, (D) $x_{\text{max}} - x_{\text{min}}$

Q 8 (D) (A) $\dfrac{dH}{dt}$ at $t = 0$, (B) H at $t = 0$, (C) H at half life

Q 9 (B) (A) wrong sign, (C) $P(2) = 0$, (D) $P(2) = 0$ and wrong sign

Q 10 (B) (A) Newton's method, (C) true for β, (D) not decreasing here

QUESTION ELEVEN (15 marks) **Marks**

(a) **2**

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}(\tfrac{x}{2})\right]_0^1 \quad ✓$$

$$= \tfrac{\pi}{6} \quad ✓$$

(b) **2**

$$\frac{d}{dx}(x \sec x) = \sec x + x \sec x \tan x \quad ✓✓$$

[(1) for each term of the product rule.]

(c) **2**

It is external division so the ratio is $(-3) : 1$. ✓

Thus $$x = \frac{1 \times 1 + (-1)(-3)}{-3+1}$$

$$= -2 \quad ✓$$

(d) **[3]**

Now $\dfrac{dy}{dx} = \dfrac{-1}{(x+2)^2}$

so at P $\dfrac{dy}{dx} = \dfrac{-1}{16}$ ✓

Thus the equation of the tangent is

$$y - \tfrac{1}{4} = \tfrac{-1}{16}(x-2)$$ ✓

or $-16y + 4 = x - 2$

so $x + 16y - 6 = 0$ (or equivalent general form.) ✓

(e) From the given substitution **[3]**

$$\int_0^1 \frac{x}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u-1}{\sqrt{u}}\,du$$ ✓

$$= \int_1^2 u^{1/2} - u^{-1/2}\,du$$

$$= \Big[\tfrac{2}{3}u^{3/2} - 2u^{1/2}\Big]_1^2$$ ✓

$$= \tfrac{4\sqrt{2}}{3} - 2\sqrt{2} - \tfrac{2}{3} + 2$$

$$= \frac{4 - 2\sqrt{2}}{3}$$ ✓

(f) The general term in the expansion is **[3]**

$$T_r = {}^9\mathrm{C}_r \times \left(\frac{3x^2}{2}\right)^{9-r} \times \left(\frac{-1}{3x}\right)^r$$

$$= {}^9\mathrm{C}_r \times (-1)^r \times (\tfrac{3}{2})^{9-r} \times (\tfrac{1}{3})^r \times x^{18-3r}\,.$$ ✓

Thus the constant term is given by

$18 - 3r = 0$

so $r = 6\,.$ ✓

Hence the constant term is

$$T_6 = {}^9\mathrm{C}_6 \times (-1)^6 \times (\tfrac{3}{2})^3 \times (\tfrac{1}{3})^6$$

$$= \tfrac{7}{18}\,.$$ ✓

Total for Question 11: 15 Marks

QUESTION TWELVE (15 marks) **Marks**

(a) (i) **[1]**

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

$$= 0\,.$$

Since $f'(x) = 0$ it follows that $f(x)$ is constant. ✓

(ii) Since it is constant, substitute any convenient value of x. When $x = 0$, **[1]**

$$f(0) = 2\cos^{-1}(0)$$
$$= \pi \quad \checkmark$$

(b) Let $f(x) = x - e^{-x}$ then **[3]**

$$f'(x) = 1 + e^{-x}\,. \quad \checkmark$$

Now $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$= 0{\cdot}5 - \frac{0{\cdot}5 - e^{-0{\cdot}5}}{1 + e^{-0{\cdot}5}} \quad \checkmark$$
$$\doteqdot 0{\cdot}566 \qquad \text{(to 3 decimal places.)} \quad \checkmark$$

As a matter of interest, the error in this answer is $< 0{\cdot}001$.

(c) (i) From the graph, **[1]**

at α $\quad \sin^2\alpha = \cos^2\alpha$

so $\quad \tan\alpha = 1 \qquad (\alpha > 0)$

thus $\quad \alpha = \frac{\pi}{4} \quad \checkmark$

(ii) **[2]**

$$\text{Area} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 x - \sin^2 x\,dx$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2x\,dx \quad \checkmark$$
$$= \left[\tfrac{1}{2}\sin 2x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= 1 \quad \checkmark$$

(d) Clearly $\triangle MXN$ is isosceles (equal radii.) **[3]**

$\angle XMN = \alpha + \beta$ (base angles of isosceles $\triangle MXN$) ✓

$\angle XYZ = \alpha$ (angle in the alternate segment) ✓

Thus $\angle MXY = \beta$ (converse of exterior angle of $\triangle MXY$) ✓

Hence $\angle MXY = \angle MXZ$ and thus MX bisects $\angle YXZ$.

(e) (i) Substitute the given function to get: **[1]**

$$\frac{dP}{dt} = -kBe^{-kt}$$
$$= k(M - M - Be^{-kt}) \quad \checkmark$$
$$= k(M - P)\,.$$

(ii) From $P(0) = 0$ we get **[1]**

$$0 = M + B\,.$$

Thus $B = -M$ ✓

and $P = M(1 - e^{-kt})$.

(iii) Substituting, **[1]**

at $t = 5$ $\quad \frac{1}{2}M = M(1 - e^{-5k})$

so $\quad e^{5k} = 2$

or $\quad k = \frac{1}{5}\log 2$ √

(iv) When $P = \frac{19}{20}M$ we get **[1]**

$$\frac{19}{20}M = M(1 - e^{-kt})$$

or $\quad \frac{19}{20} = 1 - e^{-kt}$

so $\quad e^{kt} = 20$

thus $\quad t = \frac{1}{k}\log 20$

$\quad \doteqdot 21\,\text{h}\ 37\,\text{min}$ √

Total for Question 12: 15 Marks

QUESTION THIRTEEN (15 marks)

Marks

(a) The simultaneous equations to be solved are: **[3]**

$$\alpha + \beta = \gamma \qquad [1]$$

$$\alpha + \beta + \gamma = 3 \qquad [2]$$

$$\alpha\beta\gamma = -\tfrac{3}{2} \qquad [3]$$ √

From [1] and [2]

$$2\gamma = 3$$

so $\quad \gamma = \frac{3}{2}$. √

Thus from [3] and $\alpha \times$ [1] we get

$$\alpha\beta = -1 \qquad [4]$$

and $\quad \alpha^2 + \alpha\beta = \frac{3}{2}\alpha \qquad [5]$

Combining [4] and [5] yields the quadratic equation

$$2\alpha^2 - 3\alpha - 2 = 0$$

so $\quad (2\alpha + 1)(\alpha - 2) = 0$

thus $\quad \alpha = -\frac{1}{2}$ or 2.

In either case the three roots are $-\frac{1}{2}$, $\frac{3}{2}$ and 2. √

(b) (i) Differentiating the equation of the parabola **2**

$$\frac{dy}{dx} = \tfrac{1}{2a}x$$

so at P $\quad \dfrac{dy}{dx} = p$

Hence the gradient of the normal is $-\frac{1}{p}$. ✓

Thus the equation of the normal is

$$y - ap^2 = -\tfrac{1}{p}(x - 2ap)$$ ✓

or $\quad x + py = 2ap + ap^3$.

(ii) The x-coordinates of M, P and N are **2**

$M_x = 0$, $\quad P_x = 2ap \quad$ and $\quad N_x = 2ap + ap^3$ ✓

so the ratio of the differences is

$$\frac{ap^3}{2ap} = \frac{3}{2}$$

or $\quad p^2 = 3$

so $\quad p = \sqrt{3} \qquad$ (since $p > 0$.) ✓

(c) **3**

Now $\quad c^2 = 4 + x^2 \qquad$ (by Pythagoras)

Differentiate using the chain rule to get

$$2c\,\frac{dc}{dt} = 2x\,\frac{dx}{dt}$$

thus $\quad \dfrac{dx}{dt} = \dfrac{c}{x} \times \dfrac{dc}{dt}$. ✓

At $c = 5{\cdot}2$, $\quad x = 4{\cdot}8$ ✓

thus $\quad \dfrac{dx}{dt} = \dfrac{5{\cdot}2}{4{\cdot}8} \times -0{\cdot}1$

$\quad \doteqdot -0{\cdot}11$m/s $\qquad$ (to the nearest cm/s) ✓

(d) (i) At $\alpha = 30^\circ$ the vertical equation of motion is: **2**

$$\begin{aligned} y &= -5t^2 + 6t - 1 \\ &= -(5t-1)(t-1) \end{aligned}$$

which has vertex at

$$\begin{aligned} t &= \tfrac{1}{2}(\tfrac{1}{5} + 1) \\ &= \tfrac{3}{5} \end{aligned}$$ ✓

so $\quad y_{\max} = -5 \times (\tfrac{3}{5})^2 + 6 \times \tfrac{3}{5} - 1$

$\quad = \tfrac{4}{5}$. ✓

[An alternative answer was accepted but is not included in this answer sheet.]

(ii) To exactly reach G, from the horizontal equation of motion **3**

$$t = \frac{1}{3\cos\alpha}$$

so $0 = 4\tan\alpha - \frac{5}{9}\sec^2\alpha - 1$ ✓

whence $0 = 5\sec^2\alpha - 36\tan\alpha + 9$

or $0 = 5\tan^2\alpha - 36\tan\alpha + 14\,.$ ✓

Thus $\tan\alpha = \frac{36-\sqrt{1016}}{10}$ or $\frac{36+\sqrt{1016}}{10}$

and $\alpha \doteqdot 22{\cdot}4^\circ$ or $81{\cdot}6^\circ\,.$

So in order to land at or beyond G

$$25^\circ \le \alpha \le 80^\circ\,.$$ ✓

[Also accept $20^\circ \le \alpha \le 80^\circ$, even though at the lower limit the ball does not reach the green.]

Total for Question 13: 15 Marks

QUESTION FOURTEEN (15 marks)

Marks

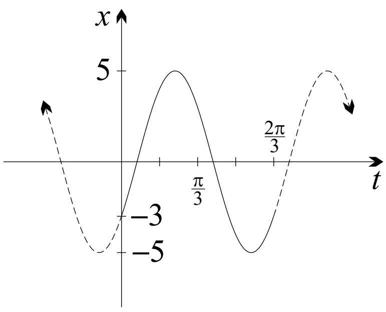
(a) (i) From the given equation and information: **1**

$$144 = n^2\left(5^2 - (-3)^2\right)$$

so $n = 3$ ✓

hence $T = \frac{2\pi}{3}\,.$

(ii) **1**



✓

[No penalty if the graph goes beyond the prescribed domain.]

(iii) At $t = 0$ it is given that **2**

$$5\sin(-\theta) = -3$$ ✓

so $\sin\theta = \frac{3}{5}$

thus $\theta \doteqdot 0{\cdot}64$ (radians, to two decimal places) ✓

(b) By the chain rule **1**

$$\frac{dy}{dx} = \frac{1}{1+(\frac{a}{x})^2} \times \frac{-a}{x^2}$$ ✓

$$= \frac{-a}{x^2+a^2}\,.$$

(c) (i) From the diagram **1**

$$\theta = \angle OAB - \angle OAC$$ ✓

$$= \tan^{-1}\left(\frac{g+h}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right).$$

(ii) From part (a) **2**

$$\frac{d\theta}{dx} = \frac{-(g+h)}{x^2+(g+h)^2} + \frac{h}{x^2+h^2}$$ ✓

so at the stationary point, where $\theta' = 0$,

$$\frac{(g+h)}{x^2+(g+h)^2} = \frac{h}{x^2+h^2}$$

or $(g+h)x^2 + (g+h)h^2 = hx^2 + h(g+h)^2$

thus $gx^2 = h(g+h)\big((g+h) - h\big)$

so $x^2 = h(g+h)$ ✓

hence $x = \sqrt{h(g+h)}$.

(iii) At $x = \sqrt{h(g+h)}$, the maximum value of θ is **2**

$$\theta_{\max} = \tan^{-1}\left(\sqrt{\frac{g+h}{h}}\right) - \tan^{-1}\left(\sqrt{\frac{h}{g+h}}\right)$$

so $$\tan\tfrac{\pi}{4} = \frac{\sqrt{\frac{g+h}{h}} - \sqrt{\frac{h}{g+h}}}{1 + \sqrt{\frac{g+h}{h}} \times \sqrt{\frac{h}{g+h}}}$$

or $$2 = \sqrt{\frac{g+h}{h}} - \sqrt{\frac{h}{g+h}}.$$ ✓

Squaring both sides yields

$$\frac{g+h}{h} - 2 + \frac{h}{g+h} = 4$$

or $$\frac{g+h}{h} + \frac{h}{g+h} = 6$$

so $(g+h)^2 + h^2 = 6h(g+h)$

and $(g+h)^2 - 6h(g+h) + 9h^2 = 8h^2$ (completing the square.)

Thus $(g-2h)^2 = 8h^2$,

so $g = 2h(1+\sqrt{2})$ or $2h(1-\sqrt{2})$. ✓

But $g > 0$, so

$$g = 2h(1+\sqrt{2}).$$

(d) (i) Using the definition: **2**

$$\begin{aligned} {}^n\mathcal{L}_1 &= {}^n\mathrm{C}_1 \times {}^{n-1}\mathrm{C}_0 \times (n-1)! \\ &= n \times 1 \times (n-1)! \\ &= n! \end{aligned}$$ √

$$\begin{aligned} {}^n\mathcal{L}_n &= {}^n\mathrm{C}_n \times {}^{n-1}\mathrm{C}_{n-1} \times (n-n)! \\ &= 1 \times 1 \times 1 \\ &= 1 \end{aligned}$$ √

(ii) There are several methods to do this. Here is one of them. **3**

$$\text{RHS} = {}^n\mathrm{C}_r \times {}^{n-1}\mathrm{C}_{r-1} \times (n-r)! \times (n+r) + {}^n\mathrm{C}_{r-1} \times {}^{n-1}\mathrm{C}_{r-2}(n-k+1)!$$

$$= \frac{n!}{(n-r)!r!} \times \frac{(n-1)!}{(n-r)!(r-1)!} \times (n-r)! \times (n+r) + \frac{n!}{(n-r+1)!(r-1)!} \times \frac{(n-1)!}{(n-r+1)!(r-2)!} \times (n-r+1)!$$

$$= \frac{n!(n-1)!}{(n-r)!(r-1)!(r-2)!} \times \left(\frac{n+r}{r(r-1)} + \frac{1}{n-r+1} \right)$$ √

$$= \frac{n!(n-1)!}{(n-r)!(r-1)!(r-2)!} \times \frac{(n+r)(n-r+1) + r(r-1)}{r(r-1)(n-r+1)}$$

$$= \frac{n!(n-1)!}{(n-r)!(r-1)!(r-2)!} \times \frac{n(n+1)}{r(r-1)(n-r+1)}$$ √

$$= \frac{(n+1)!}{(n-r+1)!r!} \times \frac{n!}{(n-r+1)!(r-1)!} \times (n-r+1)!$$

$$= {}^{n+1}\mathrm{C}_r \times {}^n\mathrm{C}_{r-1} \times (n-r+1)!$$ √

$$= {}^{n+1}\mathcal{L}_r\,.$$

Total for Question 14: 15 Marks

BR/DNW